p-Origamis: Strata, Veech Groups and Sums of Lyapunov Exponents

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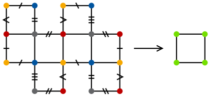
partly joint with Johannes Flake

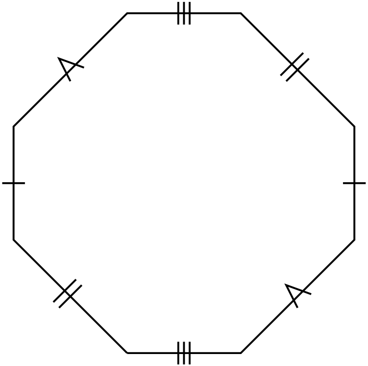
Fourth annual conference of the SFB-TRR 195
September 24, 2020



Translation surfaces and origamis

Definition

A **translation surface** is a connected surface obtained by gluing the edges of finitely many polygons by translations.



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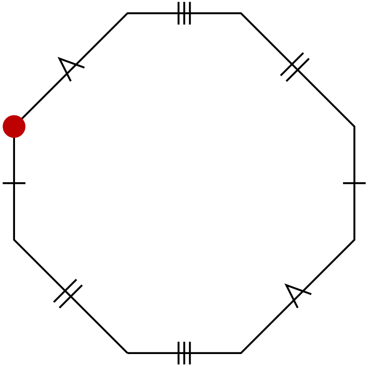

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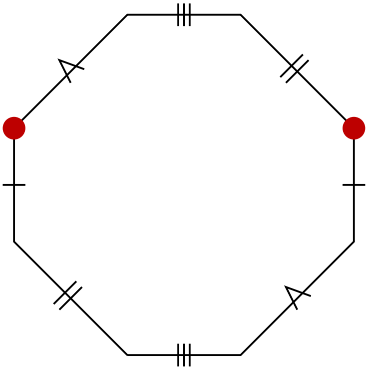

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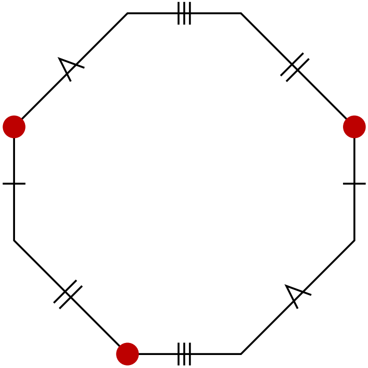

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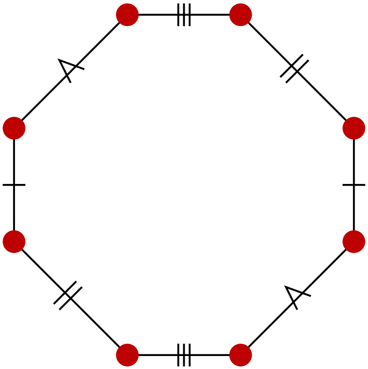

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The **stratum** $H(b_1 \times a_1, \ldots, b_m \times a_m) =: H(\underline{b} \times \underline{a})$ is defined as $\{X \text{ translation surface} \mid X \text{ has } b_i \text{ singularities of cone angle } (a_i + 1) \cdot 2\pi\}$

A **square-tiled surface** or **origami** is a translation surface obtained by gluing finitely many unit squares. The number of squares is called **degree**.

Remark:
An origami O defines a torus cover $O \to \mathbb{T}$ ramified over at most one point.

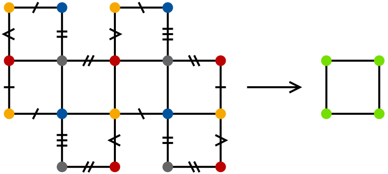
Normal origamis and p-origamis

Definition

An origami O is called **normal** if the induced cover $O \to \mathbb{T}$ is normal.

O is a p-**origami** if it is normal and its deck group is a p-group.

Example:



Normal origamis and p-origamis

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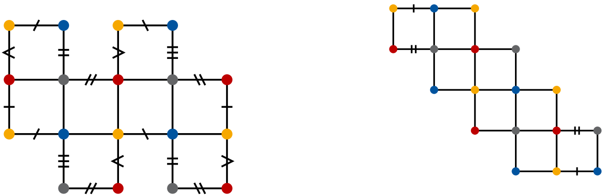
Examples:



Questions:

(A) In which strata do p-origamis occur?

(B) Do p-origamis with isomorphic deck group lie in the same stratum?

Question A: In which strata do p-origamis occur?

Theorem 1 [Flake - T., 2020]

Let $n \quad \mathbb{N}_{>2}$. For p-origamis of degree p^n and genus $g > 1$ exactly the following strata appear

$$H\left(p^{n-k} \times \left(p^k - 1\right)\right), \textit{ where}$$

$1 \leq k \leq n-2$ *if* $p = 2$ *and*
$1 \leq k \leq \frac{n-1}{2}$ *if* $p > 2$.

Lemma [Flake - T., 2020]

Let G be a finite p-group for an odd prime. Then $\exp(G)^2 < |G|$.

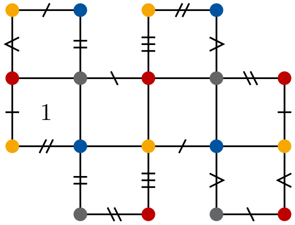
Essential concepts

Definition

An origami O is called **normal** if the induced cover $O \to \mathbb{T}$ is normal.
O is a *p*-**origami** if it is normal and its deck group is a p-group.

Remark:

- $\{$ squares in tiling $\}$ $\overset{1-1}{\longleftrightarrow}$ $\{$ elements of deck group $\}$
  The eierlegende Wollmilchsau is the 2-origami (Q_8, i, j).



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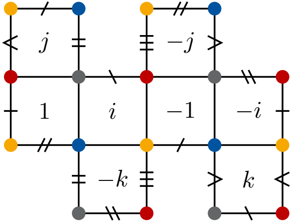

Essential concepts

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Remark:

- $\{$ squares in tiling $\}$ $\overset{1-1}{\longleftrightarrow}$ $\{$ elements of deck group $\}$
  Notation: $O = (G, x, y)$
- $[x, y]$ corresponds to "going 2π around a singularity"
  Hence: $\mathrm{ord}([x, y]) = k$ cone angle $= k \cdot 2\pi$
- All singularities of a normal origami have the same cone angle.

Question B: p-origamis with isomorphic deck groups

Normal origamis with isomorphic deck group do not lie in the same stratum, e.g., alternating group A_5:

$$(x, y) = ((1, 2, 3), (3, 4, 5)) \qquad (A_5, x, y) \quad H(20 \times 2)$$
$$(x, y) = ((3, 4, 5), (1, 3)(2, 4)) \qquad (A_5, x, y) \quad H(12 \times 4)$$

But for p-origamis they often do.

Definition

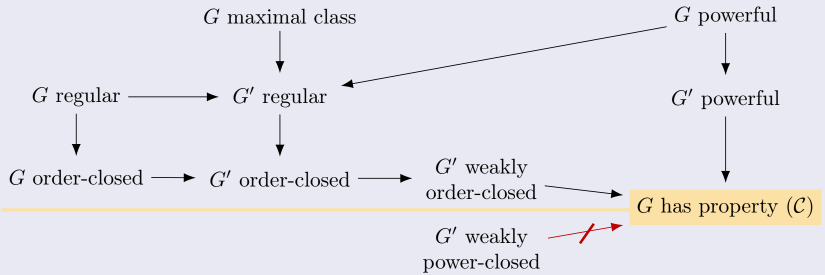
A finite 2-generated group G has **property** (C), if for all 2-generating sets $\{x, y\}$ the order of $[x, y]$ are equal.

Remark:
G has property (C) if and only if all normal origamis with deck group G lie in the same stratum.

Question B: p-origamis with isomorphic deck groups

Theorem 2 [Flake - T., 2020]

For a finite 2-generated p-group G the following implications hold

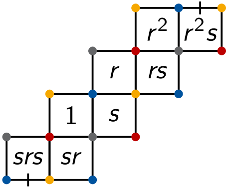


For each prime p, there is a p-group G S_{p^4} without property $(\mathcal{C})$.

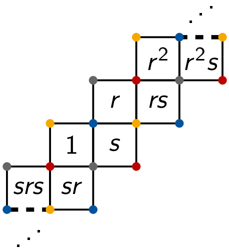
Generalization to infinite origamis and pro-p groups

Remark:
If $G = \varprojlim G_n$ is a pro-p group, then there is a correspondence between families of p-origamis with deck group G_n and infinite normal origamis.

Origami (D_4, s, sr)



Origami (D_∞, s, sr)



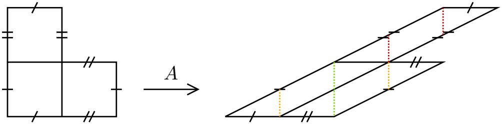
Proposition 3 [Flake - T., 2020]

Let G be a topologically 2-generated pro-p group. If G is either weakly order-closed or powerful, then G has property (C).

Veech groups

$\mathrm{SL}(2,\mathbb{R})$ acts on each stratum $H(\underline{b} \times \underline{a})$.

E.g., the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ acts as follows



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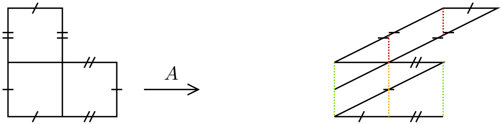

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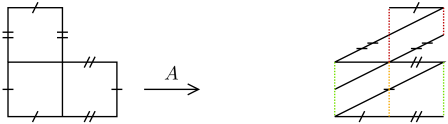

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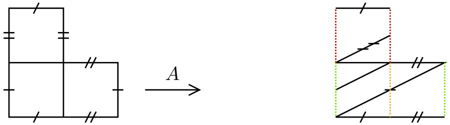

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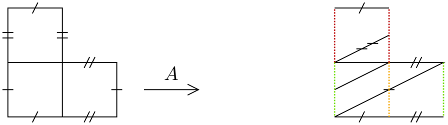


Definition

For X $H(\underline{b} \times \underline{a})$, the stabilizer $\mathrm{Stab}_{\mathrm{SL}(2,\mathbb{R})}(X) =: \mathrm{SL}(X)$ is called the **Veech group of** X.

Remark:
The group $\mathrm{SL}(X)$ detects whether X defines a Teichmüller curve.
The Veech group of an origami is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$.
For normal origamis the deck group can be used to compute the Veech group.

Results for families of group: e.g. of the form $C_{2^n} \circ C_{2^k}$

Example series	index of Veech group	congruence group of level	
$(G_{(n,k)}, r, s)$	$3 \cdot 2^{k-1}$	2^{k-1}	
$(A_{(n,1)}, r, s)$	$3 \cdot 2^{n-3}$	2^{n-2}	
$(B_{(n,1)}, r, s)$	6	2	
(W_m, x, y)	$3 \cdot 2^{m-1}$	2^m	

The sum of Lyapunov exponents

Theorem [Eskin - Kontsevich - Zorich, 2014]

The sum of nonnegative Lyapunov-exponents equals

$$\frac{1}{12}\cdot\sum_{i=1}^{m}\frac{a_i\cdot(a_i+2)}{a_i+1}+\frac{1}{\#orb}\cdot\sum_{O_i\quad orb}\ \sum_{O_i=\quad cyl_{ij}}\frac{h_{ij}}{w_{ij}},$$

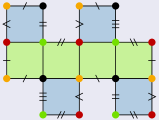
where O is an origami in the stratum $H(1\times a_1,\ldots,1\times a_m)$,

$orb=\mathrm{SL}(2,\mathbb{Z})\cdot O$,

cyl_{ij} = horizontal cylinders in O_i,

h_{ij} = height of cylinder cyl_{ij},

w_{ij} = width of cylinder cyl_{ij}.



For a normal origami of degree n this simplifies to

$$\frac{1}{12}\cdot m\cdot\frac{a_1\cdot(a_1+2)}{a_1+1}+\frac{1}{\#orb}\cdot\sum_{(G,x,y)\quad orb}\frac{n}{\mathrm{ord}(x)}\cdot\frac{1}{\mathrm{ord}(x)}.$$

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Example series	index of Veech group	congruence group of level	$\sum$ Lyapunov-exponents
$(G_{(n,k)}, r, s)$	$3 \cdot 2^{k-1}$	2^{k-1}	$\frac{1}{3} \cdot \left(2^{n+k-2} + 2^{n-k+1} + c\right)$
$(A_{(n,1)}, r, s)$	$3 \cdot 2^{n-3}$	2^{n-2}	$2^{n-3} + 1$
$(B_{(n,1)}, r, s)$	6	2	$3 \cdot 2^{n-3}$
(W_m, x, y)	$3 \cdot 2^{m-1}$	2^m	$\frac{1}{3} \cdot (2^{m-1} + 1)$

References:

[EKZ] *Sum of Lyapunov exponents of the Hodge bundle withrespect to the Teichmüller geodesic flow*, Publ. Math. Inst. Hautes Études Sci. 120, 2014.

[FT] *Strata of p-Origamis*, preprint 2020, arXiv:2003.13297.

[T] *Veech Groups and Sums of Lyapunov Exponents of 2-Origamis*, in preparation.

Thank you for your attention!