Systoles of Translation Surfaces

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joint work with

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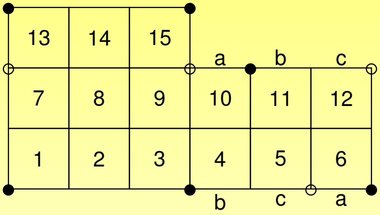
The Problem



Glue edges marked with the same label or unmarked opposite edges by translations.

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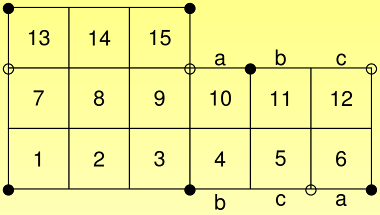


Glue edges marked with the same label or unmarked opposite edges by translations.

Goal:

Find the shortest closed geodesic on the resulting surface.

How big can it maximal be?

Translation surfaces

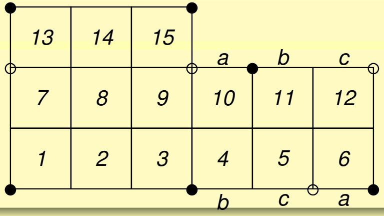
Definition

Take finitely many polygons in the Euclidean plane such that each edge has a partner edge which

- *is parallel, of same length and of opposite orientation.*

Glue each edge to its partner edge by a translation such that you obtain a closed surface. This surface is called finite translation surface with cone angle singularities.

Example

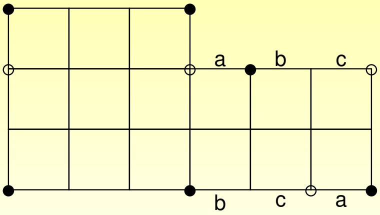


Observe: The vertices lead to cone angle singularities!

Translation surfaces

Definition

A translation atlas μ *on a surface* X *is an atlas such that all transition maps are translations.*



Translation surfaces

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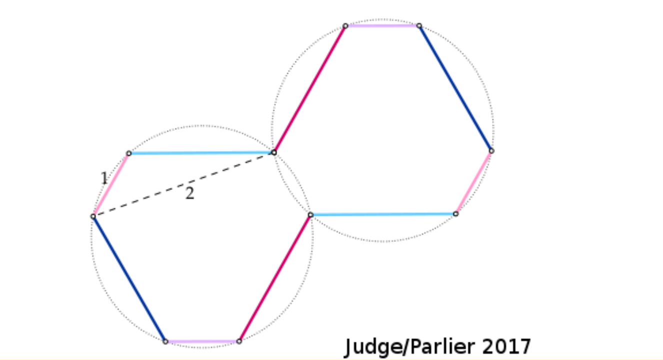

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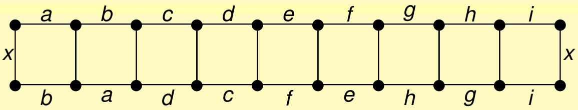


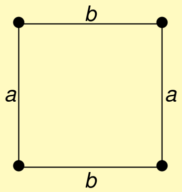
Genus 5, 1 singularity

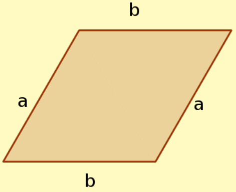
Origamis

Definition

If, in Definition 1, all polygons are copies of the unit square then the translation surface is called <u>origami</u> or <u>square-tiled surfaces</u>.

Example (Unit square torus T_0 and Hexagonal torus T_h)





Remark

Any origami is equipped with a natural covering map to the unit square torus T_0.

Strata of translation surfaces

Definition

$$\mathcal{H}_g(\alpha_1, \ldots, \alpha_k) = \left\{ (X, \mu) \middle| \begin{array}{l} X \textit{ has } k \textit{ singularities of angle} \\ (\alpha_1 + 1) \cdot 2\pi, \ldots, (\alpha_k + 1) \cdot 2\pi \end{array} \right\} / \sim$$

is called the stratum of translation surfaces of genus g with k singularities of angle $(\alpha_1 + 1)2\pi, \ldots, (\alpha_k + 1)2\pi$.

Remark

$$\alpha_1 + \ldots + \alpha_k = 2g - 2$$

Strata of translation surfaces

Remark

Strata of translation surfaces have been intensively studied since the 1980s, for example by Avila, Chen, Delecroix, Eskin, Filip, Forni, Hamenstädt, Hooper, Hubert, Judge, Kontsevich, Lanneau, Leininger, Lelièvre, Masur, Matheus, McMullen, Mirzakhani, Möller, Rafi, Smillie, Valdez, Veech, Weiss, Wright, Yoccoz and Zorich

Systolic Ratio

Definition

The systolic ratio $\mathrm{SR}(X, \mu)$ *of a translation surface* (X, μ) *is defined as:*

$$\mathrm{SR}(X, \mu) = \frac{\text{(length of shortest simple closed geodesic)}^2}{\text{area}(X, \mu)}$$

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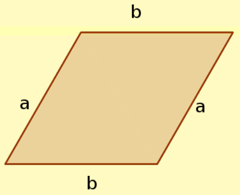

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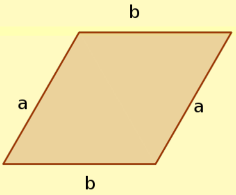

$$\mathrm{SR}(T_h) = \frac{2}{\sqrt{3}}$$

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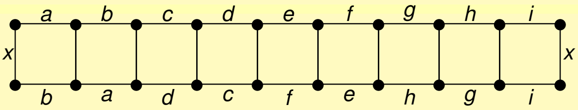

Surface with $9 = 2 \cdot 5 - 1$ *squares and one singularity in* $\mathcal{H}_5(8)$

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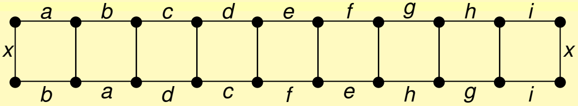

Surface with $9 = 2 \cdot 5 - 1$ *squares and one singularity in* $\mathcal{H}_5(8)$

$$\mathrm{SR}(X) = \frac{1}{9}$$

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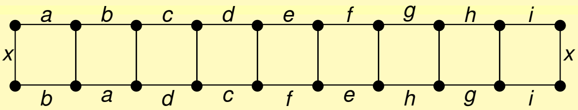

Surface with $2g - 1$ *squares and one singularity in* $\mathcal{H}_g(2g-2)$

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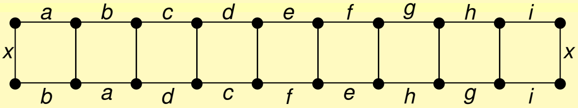

Surface with $2g-1$ *squares and one singularity in* $\mathcal{H}_g(2g-2)$

$$\mathrm{SR}(X) = \frac{1}{2g-1}$$

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  -> Yes, for genus g.

The problem

Questions:

- What is the supremal systolic ratio for translation surfaces of genus g?
- What is the supremal systolic ratio for translation surfaces in a stratum $\mathcal{H}_g(\alpha_1, \ldots, \alpha_k)$?
- Is the supremum achieved?
  - -> Yes, for genus g.
  - -> Unknown for general strata.

The stratum $\mathcal{H}_g(2g-2)$

Theorem 1 (Judge+Parlier, Boissy+Geninska)

The maximal systolic ratio in the stratum $\mathcal{H}_g(2g-2)$ is

$$\frac{2}{\sqrt{3}\cdot(2g-1)}.$$

It is achieved for the hexagonal shearing of the following origami with $2g-1$ squares.

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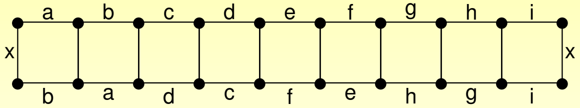

Example

For $g=2$ the maximal length of a systole in $\mathcal{H}_2(2)$ is $\frac{2}{3\sqrt{3}} \sim 0.38$

Upper bound in general strata

Theorem 2 (Columbus/Herrlich/Mützel/W)

Let X be a translation surface in the stratum $\mathcal{H}(k_1, \ldots, k_n)$, such that $k_1 \leq k_2 \leq \ldots \leq k_n$. Then

$$SR(\mathcal{H}(k_1, \ldots, k_n)) \leq \frac{4}{\pi \cdot (k_n + 1)}.$$

What about $\mathcal{H}_2(1,1)$?

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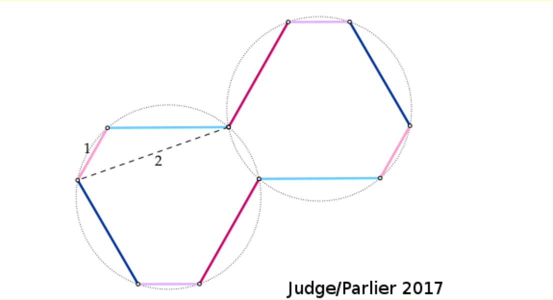
Example (Judge/Parlier)



The systolic ratio is

$$\frac{2 \cdot (\sqrt{13} - 3)^2}{\sqrt{3} \cdot (1 - \frac{3}{4}(\sqrt{13} - 3)^2)} \sim 0.58$$

What about $\mathcal{H}_2(1,1)$?

Conjecture (Judge/Parlier)

The surface on the last slide is a surface of maximal systolic ratio.

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Remark

This surface is not a sheared origami.

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There exists a surface with maximal systolic ratio in $\mathcal{H}_2(1,1)$.

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Idea:

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There is always a systole which is a concatenation of saddle connections, i.e. straight segments between singularities.

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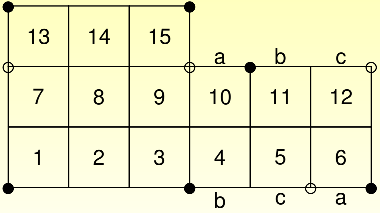
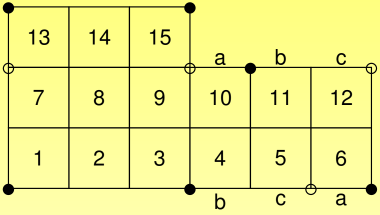
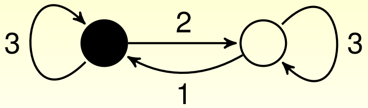


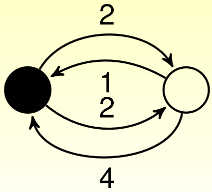
Graph of saddle connections



Ansatz:

- Consider graph of saddle connections.





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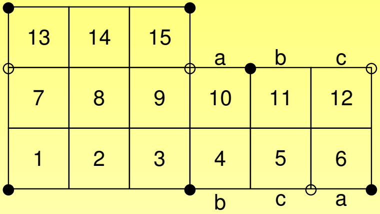

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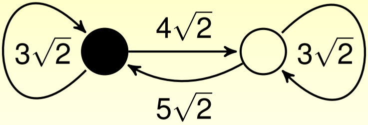

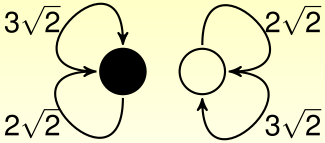

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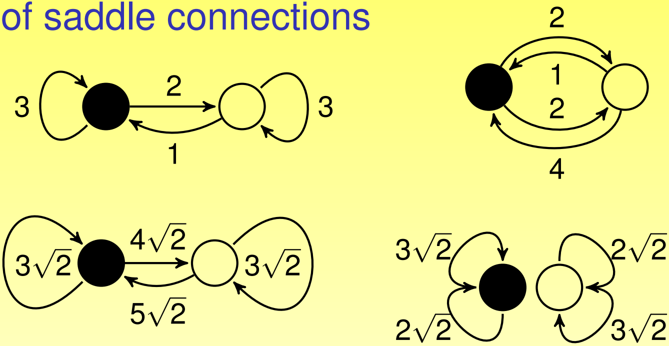

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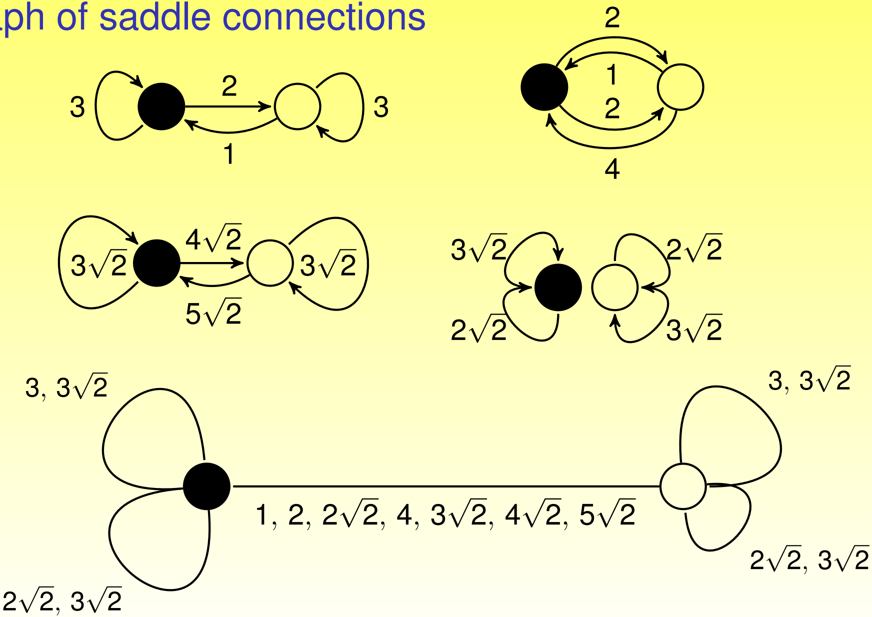


Systoles of origamis in $H_2(1,1)$

n	ℓ	ℓ^2/n
1		
2		
3		
4	$\sqrt{2}$	0.5
5	$\sqrt{2}$	0.4
6	$\sqrt{2}$	~ 0.33
7	$\sqrt{2}$	~ 0.29
8	2	0.5
9	2	~ 0.44
10	2	0.4
11	$\sqrt{5}$	~ 0.45
12	$\sqrt{5}$	~ 0.42
13	$\sqrt{5}$	~ 0.38
14	$\sqrt{5}$	~ 0.36
15	$\sqrt{5}$	~ 0.33
16	$1+\sqrt{2}$	~ 0.36

n	ℓ	ℓ^2/n
17	$2\sqrt{2}$	~ 0.47
18	$2\sqrt{2}$	~ 0.44
19	3	~ 0.47
20	$\sqrt{10}$	0.50
21	$\sqrt{10}$	~ 0.48
22	$\sqrt{10}$	~ 0.45
23	$\sqrt{10}$	~ 0.43
24	$\sqrt{13}$	~ 0.54
25	$\sqrt{13}$	~ 0.52
26	$\sqrt{13}$	0.50
27	$\sqrt{13}$	~ 0.48
28	$\sqrt{13}$	~ 0.46
29	$\sqrt{13}$	~ 0.45
30	$\sqrt{17}$	~ 0.57
31	4	~ 0.52
32	$\sqrt{17}$	~ 0.53

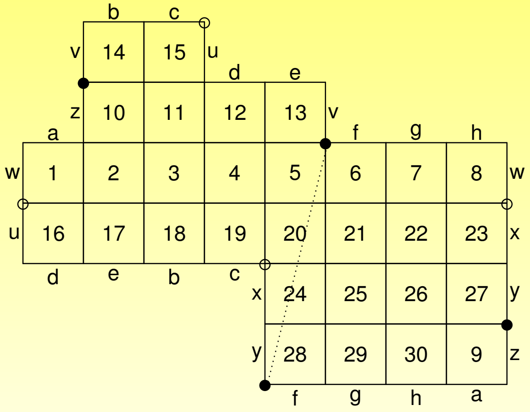
n	ℓ	ℓ^2/n
33	$\sqrt{17}$	~ 0.52
34	$\sqrt{17}$	0.50
35	$\sqrt{17}$	~ 0.49
36	$3\sqrt{2}$	0.50
37	$\sqrt{5}+2$	~ 0.48
38	$3\sqrt{2}$	~ 0.47
39	$3\sqrt{2}$	~ 0.46
40	$2\sqrt{5}$	0.5
41	$2\sqrt{5}$	~ 0.49
42	$2\sqrt{5}$	~ 0.48
43	$2\sqrt{5}$	~ 0.47
44	$2\sqrt{5}$	~ 0.45
45	$2\sqrt{5}$	~ 0.44
46	5	~ 0.54
47	5	~ 0.53
48	5	~ 0.52

Systoles of origamis in $H_2(1,1)$

n	ℓ	ℓ^2/n
49	5	~ 0.51
50	$\sqrt{26}$	0.52
51	$\sqrt{26}$	~ 0.51
52	$\sqrt{26}$	0.50
53	$1+\sqrt{17}$	~ 0.50
54	$2+\sqrt{10}$	~ 0.49
55	$1+3\sqrt{2}$	~ 0.50
56	$\sqrt{29}$	~ 0.52
57	$\sqrt{29}$	~ 0.51
58	$\sqrt{29}$	~ 0.50
59	$4+\sqrt{2}$	~ 0.50
60	$\sqrt{34}$	~ 0.57
61	$1+2\sqrt{5}$	~ 0.49
62	$4\sqrt{2}$	~ 0.52
63	$\sqrt{34}$	~ 0.54
64	$\sqrt{34}$	~ 0.53

n	ℓ	ℓ^2/n
65	$\sqrt{34}$	~ 0.52
66	$\sqrt{34}$	~ 0.52
67	$\sqrt{10}+2\sqrt{2}$	~ 0.54

An approximation origami



An origami O with 30 squares whose systole has length $\sqrt{17}$.
O has two singularities $\bullet$ and $\circ$. The dashed line is a systole.

Further goals

- Settle problem in $\mathcal{H}_2(1,1)$.
- Obtain conjectures for growth rate of the supremal systolic ratio in genus g.
- Find out whether the supremal systolic ratio is achieved for translation surfaces in the principal strata $\mathcal{H}_g(1,\ldots,1)$.
- In which stratum lies the translation surface of genus g with maximal systolic ratio.