Interpolating Partition Categories

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joint work with Johannes Flake
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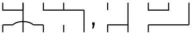
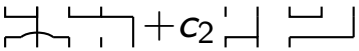
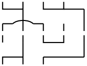
Deligne '07

$$P = \{$$  $$, \ldots\}$$

$\underline{\mathrm{Rep}}(S_n)$

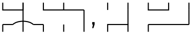
Obj.: ...

Mor.: c_1  $+ \ldots \in \mathbb{C}P$

Comp.: $n^{\# loops}$ 

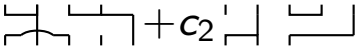
$\longrightarrow$ $\mathrm{Rep}(S_n)$

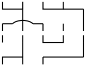
Deligne '07

$P = \{$  $, \ldots\}$

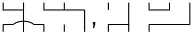
$\underline{\mathrm{Rep}}(P, t)$, $t \in \mathbb{C}$

Obj.: ...

Mor.: c_1  $+ \ldots \in \mathbb{C}P$

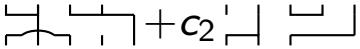
Comp.: $t^{\#loops}$ 

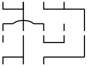
$\xrightarrow{t = n \in \mathbb{N}}$ $\mathrm{Rep}(S_n)$

$$\mathcal{C} \subseteq P = \{$$  $$, \ldots\}$$

Rep$(\mathcal{C}, t)$, $t \in \mathbb{C}$

Obj.: …

Mor.: c_1  $+ \ldots \in \mathbb{C}\mathcal{C}$

Comp.: $t^{\#loops}$ 

$t = n \in \mathbb{N}$ ⟶

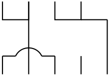
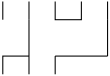
$\mathrm{Rep}(G_n(\mathcal{C}))$
representation cat. of
easy quantum group $G_n(\mathcal{C})$

Interpolation partition categories

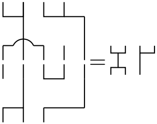
Categories of partitions

Banica – Speicher '09:

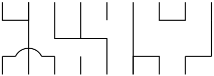
- $P(k,l) := \{\text{partitions of } \{1,\ldots,k,1',\ldots,l'\}\}$ for $k,l \in \mathbb{N}_0$.

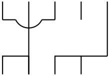
$p =$  $\in P(4,5)$, $q =$  $\in P(5,3)$

- Operations:

$q \cdot p =$ 

$l(q,p) = \#\mathit{loops} = 1$

$p \otimes q =$ 

$p^* =$ 

- $\mathcal{C} \subseteq \bigcup_{k,l\in\mathbb{N}_0} P(k,l)$ *category of partitions* if $\mathcal{C}$ is closed under the operations and $|$, ⊓ $\in \mathcal{C}$

Interpolating partition categories

$\mathcal{C}$ category of partitions, $t \in \mathbb{C}$

$\underline{\mathrm{Rep}}(\mathcal{C}, t)$ = Karoubi envelope of

Obj.:	$[k], k \in \mathbb{N}_0$,
Mor.:	$\mathrm{Hom}([k],[l]) = \mathbb{C}\mathcal{C}(k,l)$,
Comp.:	$\mathrm{Hom}([l],[m]) \times \mathrm{Hom}([k],[l]) \to \mathrm{Hom}([k],[m])$,
	$(q,p) \mapsto qp := t^{l(q,p)}\ q \cdot p$

$\underline{\mathrm{Rep}}(\mathcal{C}, t)$ is a rigid monoidal *-category.

Monoidal structure:	$[k] \otimes [l] = [k+l], p \otimes q$
*-structure:	$[k]^* = [k], p^*$
Duals:	$[k]^\vee = [k]$

Interpolating representation categories

$\mathrm{Rep}(S_n)$

Obj.: $v{:}S_n{\to}\mathrm{GL}(V)$

Mor.: $\mathrm{Hom}_{S_n}(V,W)$

$u : S_n \to \mathrm{GL}_n(\mathbb{C})$ canonical representation of S_n

$\mathrm{Rep}(S_n)$

Karoubi env. of

Obj.: $u^{\otimes k}, k{\in}\mathbb{N}_0$

Mor.: $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k},(\mathbb{C}^n)^{\otimes l})$

Interpolating representation categories

$u : S_n \to \mathrm{GL}_n(\mathbb{C})$ canonical representation of S_n

There exists an essentially surjective and full functor

$$\mathcal{F} : \underline{\mathrm{Rep}}(P, n) \to \mathrm{Rep}(S_n) \text{ with } \mathcal{F}([k]) = u^{\otimes k}.$$

$\underline{\mathrm{Rep}}(P,n)$

Karoubi env. of

Obj.: $[k], k \in \mathbb{N}_0$

Mor.: $\mathrm{Hom}([k],[l]) = \mathbb{C}P(k,l)$

$\xrightarrow{\mathcal{F}}$

$\mathrm{Rep}(S_n)$

Karoubi env. of

Obj.: $u^{\otimes k}, k \in \mathbb{N}_0$

Mor.: $\mathrm{Hom}_{S_n}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$

Interpolating representation categories

1. $G \subseteq \mathrm{GL}_n(\mathbb{C})$ compact matrix group
   $\rightsquigarrow$ $C(G) = \{f : G \to \mathbb{C} \text{ continuous}\}$
   commutative C*-algebra with comultiplication
   $\rightsquigarrow$ canonical representation $u : G \to \mathrm{GL}_n(\mathbb{C}) \in C(G)^{n\times n}$
2. Compact matrix quantum group (Woronowicz '87)
   $\rightsquigarrow$ ~~commutative~~ C*-algebra A with comultiplication
   $\rightsquigarrow$ 'fundamental representation' $u \in A^{n\times n}$
3. Tannaka-Krein duality (Woronowicz '88)
   Recover compact matrix quantum group G from its representation category $\mathrm{Rep}(G)$
4. Easy quantum group (Banica – Speicher '09)
   Compact matrix quantum group such that $\mathrm{Rep}(G)$ can be described by a category of partitions $\mathcal{C}$

Interpolating representation categories

There exists an essentially surjective and full functor

$$\mathcal{F} : \underline{\mathrm{Rep}}(\mathcal{C}, n) \to \mathrm{Rep}(G_n(\mathcal{C})) \text{ with } \mathcal{F}([k]) = u^{\otimes k}.$$

$\underline{\mathrm{Rep}}(\mathcal{C},n)$	$\xrightarrow{\mathcal{F}}$	$\mathrm{Rep}(G_n(\mathcal{C}))$
Karoubi env. of		Karoubi env. of
Obj.: $[k], k \in \mathbb{N}_0$		Obj.: $u^{\otimes k}, k \in \mathbb{N}_0$
Mor.: $\mathrm{Hom}([k],[l]) = \mathbb{C}\mathcal{C}(k,l)$		Mor.: $\mathrm{Hom}_{G_n(\mathcal{C})}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$

Examples

$\mathcal{C}$	$G_n(\mathcal{C})$	$\underline{\mathrm{Rep}}(\mathcal{C}, t)$	Reference
all partitions	S_n	$\underline{\mathrm{Rep}}(S_t)$	Deligne '07
pair partitions	O_n	$\underline{\mathrm{Rep}}(O_t)$	Deligne '90
non-crossing pair partitions	O_n^+	TL_q, $q = t + t^{-1}$	Graham–Lehrer '98

Semisimplicity

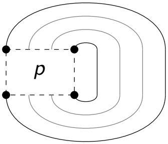
Compute $t \in \mathbb{C}$ with $\underline{\mathrm{Rep}}(\mathcal{C}, t)$ semisimple

Example

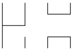
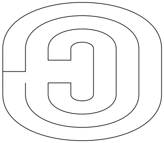
$\mathcal{C}$	$\underline{\mathrm{Rep}}(\mathcal{C}, t)$	Semisimple	Reference
all partitions	$\underline{\mathrm{Rep}}(S_t)$	$t \notin \mathbb{N}_0$	Deligne '07
pair partitions	$\underline{\mathrm{Rep}}(O_t)$	$t \notin \mathbb{Z}$	Deligne '90
non-crossing pair partitions	TL_q, $q = t + t^{-1}$	$q \neq e^{\frac{j\pi}{l}}$, $l \in \mathbb{N}_{\geq 2}$ $j \in \mathbb{N}_{\leq l-1}$	Graham–Lehrer '98

Compute $t \in \mathbb{C}$ with $\underline{\mathrm{Rep}}(\mathcal{C}, t)$ semisimple

$\underline{\mathrm{Rep}}(\mathcal{C}, t)$ is a spherical category with trace

$\mathrm{tr}(p) =$ 

Example

$\mathrm{tr}($  $) =$  $= t^2$

Compute $t \in \mathbb{C}$ with $\underline{\mathrm{Rep}}(\mathcal{C}, t)$ semisimple

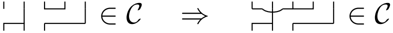
'Symmetric' trace form

$$<,>: \mathcal{C}(k,l) \times \mathcal{C}(l,k) \to \mathbb{C}, (p,q) \mapsto \mathrm{tr}(qp)$$

Etingof – Ostrik '18:
$\underline{\mathrm{Rep}}(\mathcal{C}, t)$ semisimple $\Leftrightarrow <,>$ is non-degenerate for all $k, l \in \mathbb{N}_0$

$\rightsquigarrow$ Compute determinants of the corresponding Gram matrices
$\rightsquigarrow \{t \in \mathbb{C} \mid \underline{\mathrm{Rep}}(\mathcal{C}, t)$ not semisimple$\}$ countable

Consider group-theoretical category of partitions $\mathcal{C}$, i.e. closed under coarsening the block structure.



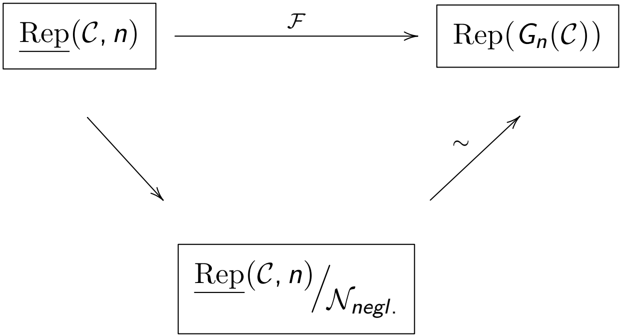
Theorem (Flake–M. '20)

$\mathcal{C}$ group-theoretical category of partitions

$$\underline{\mathrm{Rep}}(\mathcal{C}, t) \text{ semisimple } \Leftrightarrow t \notin \mathbb{N}_0$$

Semisimplification

$$\mathcal{N}_{negl.} = \{f : X \to Y \mid < f, g >= 0 \ \ \forall\, g : Y \to X\}$$



Indecomposable Objects

Indecomposable Objects

Example

$\mathcal{C}$	$\underline{\mathrm{Rep}}(\mathcal{C}, t)$	Indecomp. Obj.	Reference
all partitions	$\underline{\mathrm{Rep}}(S_t)$	Young diagrams of arbitrary size	Deligne '07
pair partitions	$\underline{\mathrm{Rep}}(O_t)$	Young diagrams of arbitrary size	Deligne '90
non-crossing pair partitions	TL_q, $q = t + t^{-1}$	Jones–Wenzl idempotents	Jones '83, Wenzl '87

Indecomposable Objects

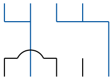
Definition

A block of a partition $p \in P(k,l)$ is called *through-block* if it contains upper points as well as lower points.

$$t(p) = \#\text{through-blocks of } p$$

Example

$p =$



$, \quad t(p) = 2$

Indecomposable Objects

Freslon – Weber '17:

Projective partitions $\mathrm{Proj}_{\mathcal{C}}$ with an equivalence relation $\sim$

$\rightsquigarrow$ associate a finite group $S(p) \leq S_{t(p)}$ to any $[p] \in \mathrm{Proj}_{\mathcal{C}}/\sim$

Theorem (Flake–M. '20)

Let $t \neq 0$. We have a bijection

$$L : \bigsqcup_{[p] \in \mathrm{Proj}_{\mathcal{C}}/\sim} \mathrm{Irr}(S(p)) \longleftrightarrow \left\{ \begin{array}{c} \textit{isom. classes of non-zero} \\ \textit{indecomp. obj. in } \underline{\mathrm{Rep}}(\mathcal{C}, t) \end{array} \right\}$$

Example

Consider $\underline{\mathrm{Rep}}(S_t) = \underline{\mathrm{Rep}}(P, t)$. Then

$\mathcal{P}/\sim = \{\emptyset, |, ||, |||, \ldots\} = \{\mathrm{id}_n \mid n \in \mathbb{N}_0\}$ and $S(\mathrm{id}_n) = S_n$.

$$\left\{ \begin{array}{c} \text{Young diagrams of} \\ \text{arbitrary size} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isom. classes of non-zero} \\ \text{indecomp. obj. in } \underline{\mathrm{Rep}}(\mathcal{C}, t) \end{array} \right\}$$

Indecomposable Objects

Remark

$K(\mathcal{C}, t) :=$ split Grothendieck ring of $\underline{\mathrm{Rep}}(\mathcal{C}, t)$
$\rightsquigarrow t(p)$ induces a $\mathbb{N}_0$-filtration on $K(\mathcal{C}, t)$

$K(S(p)) :=$ Grothendieck group of $\mathrm{Rep}(S(p))$

$$R := \bigoplus_{[p] \in \mathcal{P}/\sim} K(S(p))$$

ring with multiplication

$$[\rho_1] \cdot [\rho_2] := [\mathrm{Ind}_{S(p) \times S(q)}^{S(p \otimes q)}(\rho_1 \boxtimes \rho_2)]$$

Proposition (Flake–M. '20)

Let $t \neq 0$. There exists a ring isomorphism $R \to \mathrm{gr}\, K(\mathcal{C}, t)$ into the associated graded ring of $K(\mathcal{C}, t)$.