How to teach a computer the Elias-Williamson graphical calculus?

Lars Thorge Jensen

Université Clermont Auvergne

September 28, 2020

Table of contents

Recent new Approach

Long-standing Open Problem

Determine the decomposition numbers for symmetric groups S_k *for all* $k \geqslant 1$.

Long-standing Open Problem

Determine the characters of indecomp. tilting modules for GL_n *for all* $n \geqslant 1$.

Recent new Approach

Long-standing Open Problem

Determine the decomposition numbers for symmetric groups S_k *for all* $k \geqslant 1$.

$\Updownarrow$ Work by Donkin, Erdmann

Long-standing Open Problem

Determine the characters of indecomp. tilting modules for GL_n *for all* $n \geqslant 1$.

Recent new Approach

Long-standing Open Problem

Determine the decomposition numbers for symmetric groups S_k for all $k \geqslant 1$.

$\Updownarrow$ Work by Donkin, Erdmann

Long-standing Open Problem

Determine the characters of indecomp. tilting modules for GL_n for all $n \geqslant 1$.

Theorem (Achar-Makisumi-Riche-Williamson)

The tilting characters of a reductive algebraic group can be expressed in terms of certain p-Kazhdan-Lusztig polynomials.

Table of contents

Some Notation

- $k = \overline{k}$ of characteristic p,
- $G \supseteq B \supseteq T$ a reductive group $/k$ with Borel and split maximal torus,
- W the affine Weyl group $W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$ viewed as a Coxeter system (W, S),

Some Notation

- $k = \overline{k}$ of characteristic p,
- $G \supseteq B \supseteq T$ a reductive group $/k$ with Borel and split maximal torus,
- W the affine Weyl group $W_{\mathrm{f}} \ltimes \mathbb{Z}\Phi$ viewed as a Coxeter system (W, S),
- $\mathcal{H}$ the Hecke algebra assoc. to (W, S) over $\mathbb{Z}[v, v^{-1}]$:

$$\mathbb{Z}W \overset{\text{deform}}{\rightsquigarrow} \mathcal{H} = \bigoplus_{w \in W} \mathbb{Z}[v, v^{-1}] H_w$$

  subject to the quadratic relation $H_s^2 = (v^{-1} - v)H_s + 1$ for $s \in S$,
- ${}^k\mathbf{H}$ the *diagrammatic category of Soergel bimodules* (defined using a realization of W over k).

Origin of the p-Kazhdan-Lusztig Basis

$$\begin{tikzcd}
{}^k\mathbf{H} \arrow[d, squiggly, "\text{Decategorification}"'] \\
[{}^k\mathbf{H}]_\oplus
\end{tikzcd}$$

Theorem (Elias-Williamson)

1. *${}^k\mathbf{H}$ is a graded, monoidal Krull-Schmidt category.*

Origin of the p-Kazhdan-Lusztig Basis

$$\begin{tikzcd} {}^k\mathbf{H} \arrow[d, squiggly, "\text{Decategorification}"'] & \\ \left[{}^k\mathbf{H}\right]_\oplus \arrow[r, "\cong"] & \mathcal{H} \end{tikzcd}$$

Theorem (Elias-Williamson)

1. ${}^k\mathbf{H}$ *is a graded, monoidal Krull-Schmidt category.*
2. *There is the following* $\underbrace{\textit{isomorphism of } \mathbb{Z}[v, v^{-1}]\textit{-algebras}}$

Origin of the p-Kazhdan-Lusztig Basis

$$\begin{tikzcd} {}^{k}\mathbf{H} \arrow[d, rightsquigarrow, "\text{Decategorification}"'] & \\ {[{}^{k}\mathbf{H}]_{\oplus}} \arrow[r, "\cong"] & \mathcal{H} \end{tikzcd}$$

Theorem (Elias-Williamson)

1. *${}^{k}\mathbf{H}$ is a graded, monoidal Krull-Schmidt category.*
2. *There is the following $\underbrace{\textit{isomorphism of } \mathbb{Z}[v, v^{-1}]\textit{-algebras}}$*
3. *We have a bijection:*

$$W \overset{\sim}{\longleftrightarrow} \left\{ \substack{\textit{indecomp. objects} \\ \textit{in } {}^{k}\mathbf{H}} \right\} / \cong, (-)$$
$$w \longleftrightarrow [B_w]$$

Origin of the p-Kazhdan-Lusztig Basis

$$\begin{tikzcd} {}^k\mathbf{H} \arrow[d, rightsquigarrow, "\text{Decategorification}"'] & \\ {[{}^k\mathbf{H}]_\oplus} \arrow[r, "\cong"] & \mathcal{H} \\ {[B_w]} \arrow[r, mapsto] & {}^p\underline{H}_w \end{tikzcd}$$

Theorem (Elias-Williamson)

1. *${}^k\mathbf{H}$ is a graded, monoidal Krull-Schmidt category.*
2. *There is the following isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras*
3. *We have a bijection:*

$$W \xleftrightarrow{\sim} \left\{ \substack{\textit{indecomp. objects} \\ \textit{in } {}^k\mathbf{H}} \right\} / \cong, (-)$$
$$w \longleftrightarrow [B_w]$$

Properties of the p-Kazhdan-Lusztig Basis

Recall: $p = \mathrm{char}(k)$

Definition

$\{{}^p\underline{H}_w \mid w \in W\}$ is called the *p-canonical* or *p-Kazhdan-Lusztig basis* of $\mathcal{H}$.

Properties of the p-Kazhdan-Lusztig Basis

Recall: $p = \mathrm{char}(k)$

Definition

$\{{}^p\underline{H}_w \mid w \in W\}$ is called the *p-canonical* or *p-Kazhdan-Lusztig basis* of $\mathcal{H}$.

Properties

- (Soergel's conjecture, proved by Elias-Williamson) $\mathrm{char}(k) = 0$
  $\Longrightarrow$ p-Kazhdan-Lusztig basis = Kazhdan-Lusztig (KL) basis

Properties of the p-Kazhdan-Lusztig Basis

Recall: $p = \mathrm{char}(k)$

Definition

$\{{}^p\underline{H}_w \mid w \in W\}$ is called the *p-canonical* or *p-Kazhdan-Lusztig basis* of $\mathcal{H}$.

Properties

- (Soergel's conjecture, proved by Elias-Williamson) $\mathrm{char}(k) = 0$ $\implies$ p-Kazhdan-Lusztig basis = Kazhdan-Lusztig (KL) basis
- strong *positivity properties* (similar to the KL positivity conjectures)
- **difficult** to calculate (categorical input needed!)

p-Kazhdan-Lusztig Basis in Type $\widetilde{A_1}$ for $p = 3$

In this case: $W = \langle s, t \mid s^2 = \mathrm{Id} = t^2 \rangle$ infinite dihedral group

p-Kazhdan-Lusztig Basis in Type $\widetilde{A_1}$ for $p = 3$

In this case: $W = \langle s, t \mid s^2 = \mathrm{Id} = t^2 \rangle$ infinite dihedral group

Compare the 3-Kazhdan-Lusztig basis in terms of the KL-basis (left) and the multiplicity of $\nabla(m)$ in $T(n)$ for $SL_2(\overline{\mathbb{F}_3})$ (right):

$$
\begin{aligned}
{}^3\underline{H}_s &= \underline{H}_s \\
{}^3\underline{H}_{st} &= \underline{H}_{st} \\
{}^3\underline{H}_{sts} &= \underline{H}_{sts} \\
{}^3\underline{H}_{stst} &= \underline{H}_{st} + \underline{H}_{stst} \\
{}^3\underline{H}_{ststs} &= \underline{H}_s + \underline{H}_{ststs} \\
{}^3\underline{H}_{ststst} &= \underline{H}_{ststst} \\
{}^3\underline{H}_{stststs} &= \underline{H}_{ststs} + \underline{H}_{stststs} \\
{}^3\underline{H}_{stststst} &= \underline{H}_{stst} + \underline{H}_{stststst}
\end{aligned}
$$

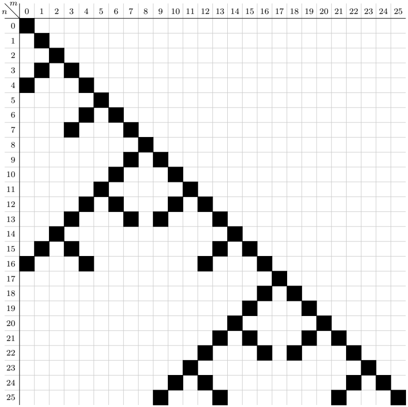


Table of contents

Splitting Bott-Samelson Bimodules

Inductively:
For $s \in S$ we have $B_s \in {}^k\mathbf{H}$ with character $[B_s] = \underline{H}_s = H_s + v$ (for all p).

Splitting Bott-Samelson Bimodules

Inductively:
For $s \in S$ we have $B_s \in {}^k\mathbf{H}$ with character $[B_s] = \underline{H}_s = H_s + v$ (for all p).

Theorem (Elias-Williamson)

For all $w \in W$ the indecomposable object B_w occurs as direct summand in

$$BS(\underline{w}) = B_{s_1} \otimes_R B_{s_2} \otimes_R \cdots \otimes_R B_{s_l}$$

for any reduced expression $\underline{w} = s_1 s_2 \dots s_l$ and not in $BS(\underline{v})$ for $v < w$.

First Idea: Splitting using Hom-Pairings

For any reduced expression $\underline{w}$ of w we have:

$$BS(\underline{w}) = B_w \oplus \underbrace{\bigoplus_{\substack{v<w \\ r\in\mathbb{Z}}} (B_v(r))^{\oplus m_{v,r}}}_{\text{Known by induction} \rightsquigarrow \text{Split off to isolate } B_w!}$$

First Idea: Splitting using Hom-Pairings

For any reduced expression $\underline{w}$ of w we have:

$$BS(\underline{w}) = B_w \oplus \underbrace{\bigoplus_{\substack{v<w \\ r\in\mathbb{Z}}} (B_v(r))^{\oplus m_{v,r}}}_{\text{Known by induction} \rightsquigarrow \text{Split off to isolate } B_w!}$$

We need a graded version of:

Lemma

Let $\mathbf{C}$ *be a* k*-linear Krull-Schmidt category and* $x \in \mathbf{C}$ *an indecomposable object with* $\mathrm{End}_{\mathbf{C}}(x) = k$. *The multiplicity of* x *in an object* $a \in \mathbf{C}$ *is given by the rank of the pairing:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(x,a) \times \mathrm{Hom}_{\mathbf{C}}(a,x) & \longrightarrow & \mathrm{End}_{\mathbf{C}}(x) = k \\ (f,g) & \longmapsto & g \circ f \end{array}$$

Light Leaves

Theorem (Libedinsky, Elias-Williamson)

The Hom*-spaces in* ${}^k\mathbf{H}$ *are free graded modules over* $R = k[\alpha_s \mid s \in S]$*.*
Libedinsky explicitly constructs a basis called light leaves.

Light Leaves

Theorem (Libedinsky, Elias-Williamson)

The Hom-*spaces in* ${}^k\mathbf{H}$ *are free graded modules over* $R = k[\alpha_s \mid s \in S]$. *Libedinsky explicitly constructs a basis called light leaves.*

Fact

The matrix representing the pairing has entries in $\mathbb{Z}$ *and full rank over* $\mathbb{Q}$ *(by the Decomposition Theorem).*

$\Longrightarrow$ Work over $k = \mathbb{Q}$ and analyze when the rank drops!

Light Leaves

Theorem (Libedinsky, Elias-Williamson)

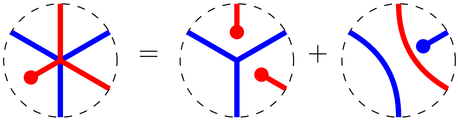
The Hom-*spaces in* ${}^k\mathbf{H}$ *are free graded modules over* $R = k[\alpha_s \mid s \in S]$. *Libedinsky explicitly constructs a basis called light leaves.*

Fact

The matrix representing the pairing has entries in $\mathbb{Z}$ *and full rank over* $\mathbb{Q}$ *(by the Decomposition Theorem).*

$\implies$ Work over $k = \mathbb{Q}$ and analyze when the rank drops!

Next Problem: How to work with Soergel diagrams in a computer?



Second Idea: Localization

Extend scalars to $Q = R[\alpha_s^{-1} \mid s \in S]$
$\implies {}^k\mathbf{H} \otimes_R Q$ is a semisimple category with simple objects $\{Q_w \mid w \in W\}$

Second Idea: Localization

Extend scalars to $Q = R[\alpha_s^{-1} \mid s \in S]$
$\Longrightarrow {}^k\mathbf{H} \otimes_R Q$ is a semisimple category with simple objects $\{Q_w \mid w \in W\}$

We have the following short exact sequences of graded R-bimodules for $s \in S$:

$$\begin{tikzcd}
0 \arrow[r] & R_s(-1) \arrow[r] & B_s \arrow[r] \arrow[d, equal] & R(1) \arrow[r] & 0 \\
0 \arrow[r] & R(-1) \arrow[r] \arrow[urr, dashed, "\cdot\alpha_s"'] & B_s \arrow[r] & R_s(1) \arrow[r] & 0
\end{tikzcd}$$

Second Idea: Localization

Extend scalars to $Q = R[\alpha_s^{-1} \mid s \in S]$
$\implies {}^k\mathbf{H} \otimes_R Q$ is a semisimple category with simple objects $\{Q_w \mid w \in W\}$

We have the following short exact sequences of graded R-bimodules for $s \in S$:

$$\begin{tikzcd}
0 \arrow[r] & R_s(-1) \arrow[r] & B_s \arrow[r] \arrow[d, equal] & R(1) \arrow[r] & 0 \\
0 \arrow[r] & R(-1) \arrow[r] \arrow[urr, dashed, "\cdot\alpha_s"] & B_s \arrow[r] & R_s(1) \arrow[r] & 0
\end{tikzcd}$$

$$\implies \quad B_s \otimes_R Q = Q_s \oplus Q \quad \text{(grading is smashed!)}$$

Soergel diagrams for Computers

In other words: $B_w \otimes_R Q = \bigoplus_{v \leqslant w} Q_v^{\oplus n_v} \implies$ Easy to encode!

After fixing an ordering, any Soergel diagram can be encoded as a matrix with entries in Q. $\implies$ Linear Algebra!

Only problem: The number of summands of $BS(\underline{w})$ of the form Q_v is $2^{l(\underline{w})}$.

Soergel diagrams for Computers

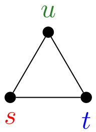
In other words: $B_w \otimes_R Q = \bigoplus_{v \leqslant w} Q_v^{\oplus n_v} \implies$ Easy to encode!

After fixing an ordering, any Soergel diagram can be encoded as a matrix with entries in Q. $\implies$ Linear Algebra!

Only problem: The number of summands of $BS(\underline{w})$ of the form Q_v is $2^{l(\underline{w})}$.

Solution: Let $s \in S$ such that $ws < w$. Then B_w inside $BS(\underline{w})$ is actually spawned by $B_{ws} \otimes_R B_s$ which is much smaller!

Illustration in $\widetilde{A}_2$



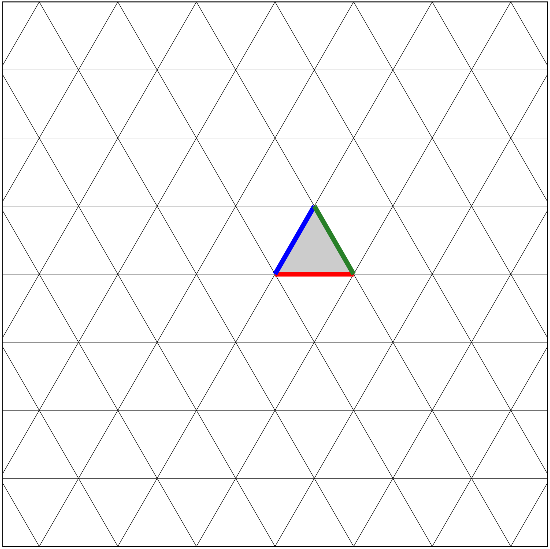
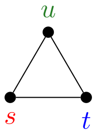


Illustration in $\widetilde{A}_2$

${}^p\underline{H}_u$



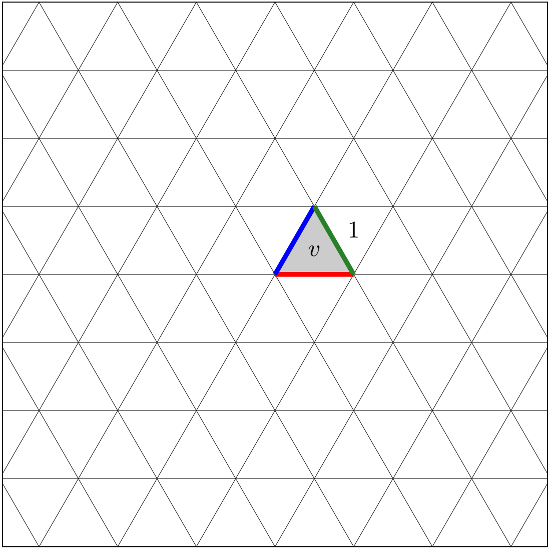
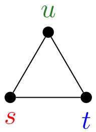


Illustration in $\widetilde{A}_2$

$${}^p\underline{H}_u \underline{H}_t = {}^p\underline{H}_{ut}$$



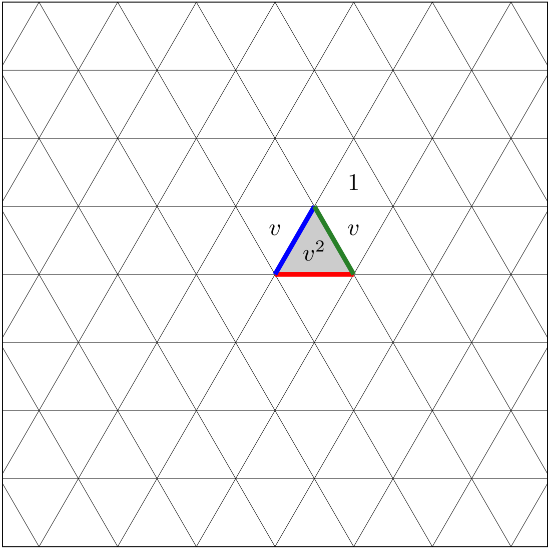
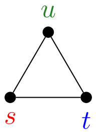


Illustration in $\widetilde{A}_2$

$$ {}^p\underline{H}_{ut}\underline{H}_s = {}^p\underline{H}_{uts} $$



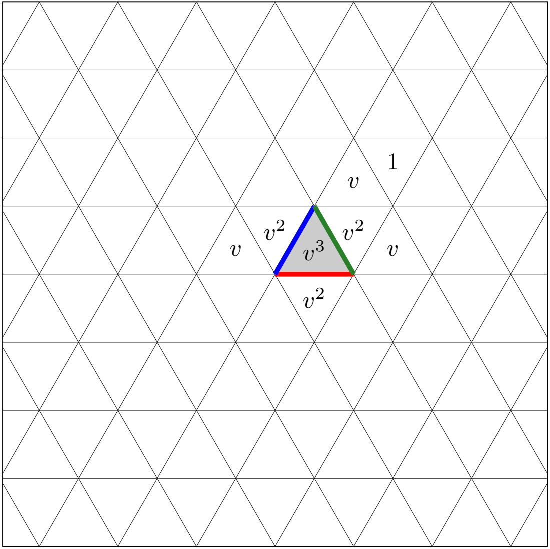
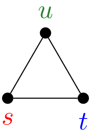


Illustration in $\widetilde{A}_2$

$${}^p\underline{H}_{uts}\underline{H}_t$$

$\rightsquigarrow$ Calculate pairing to determine ${}^p\underline{H}_{utst}$: (-1)



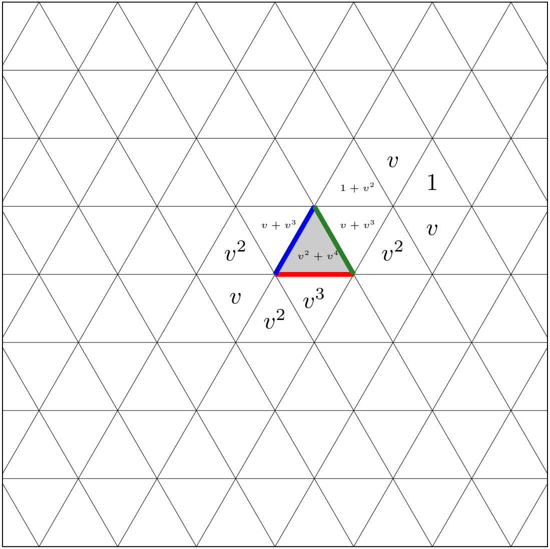
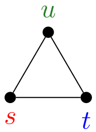
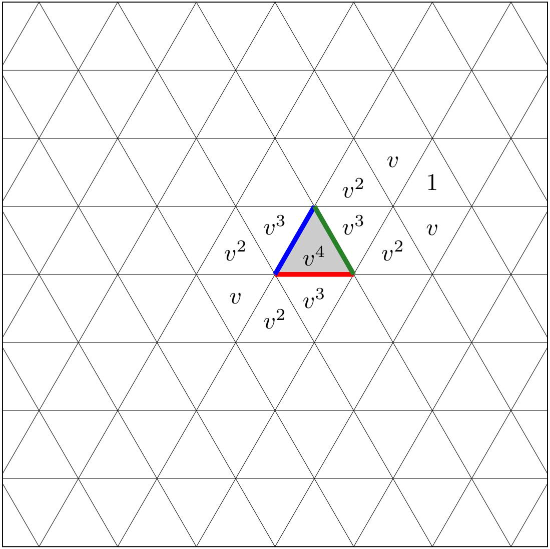


Illustration in $\widetilde{A}_2$

$$ {}^p\underline{H}_{uts}\underline{H}_t - {}^p\underline{H}_{ut} = {}^p\underline{H}_{utst} $$

independent of p!





Biggest Problem

Open Problem

While translating bases of Hom-*spaces along, rational coefficients blow up!*

Biggest Problem

Open Problem

While translating bases of Hom*-spaces along, rational coefficients blow up!*

Solution: ?
Any ideas or suggestions are welcome!

Thank you for your attention!

References I

B. Elias and G. Williamson,
Soergel Calculus
Rep. Theory (20), 2016, pp. 295–374.

B. Elias and G. Williamson,
The Hodge theory of Soergel bimodules
Ann. of Math. (2) 180.3, 2014, pp. 1089–1136.

Lars Thorge Jensen and Geordie Williamson,
The p-Canonical Basis for Hecke Algebras
Categorification and higher representation theory. Vol. 683. Contemp. Math. Amer. Math. Soc., Providence, RI, 2017, pp. 333–361.

David Kazhdan and George Lusztig,
Representations of Coxeter groups and Hecke algebras
Invent. Math. **53**, 1979, no. 2, 165–184.

W. Soergel,
Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen
J. Inst. Math. Jussieu 6.3, 2007, pp. 501–525.