Equations for the K3-Lehmer map

Simon Brandhorst,
joint work with Noam D. Elkies

September 21, 2020

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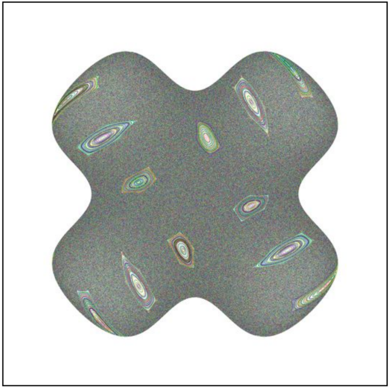
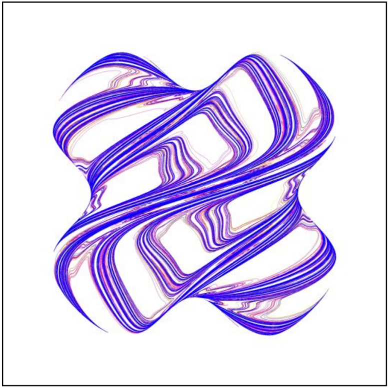
A real (K3) surface of bidegree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$





Left: some orbits Right: a stable manifold [1]

[1] Cantat, Dynamics of automorphisms of compact complex surfaces, Frontiers in Complex Dynamics, Princeton University Press, pp. 463-514.

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$\Rightarrow$ $\lambda(f)$ a Salem number (an algebraic integer)

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If equality holds, then X must be rational or a K3 surface.

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$$\begin{aligned} X_{10}/\mathbb{F}_{29} \colon y^2 &= x^3 + 19x + 19t^7 + 15 \\ x(t) &= t^4 + 7t^3 + 7t^2 + 27t + 16 \\ y(t) &= t^6 + 25t^5 + 18t^4 + 25t^3 + 15t^2 + 20t + 23 \end{aligned}$$

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Solution

Let F' be the fiber of another fibration with $F'.F = 2$. Then $F'^{\perp}/F'$ and $F^{\perp}/F$ are neighboring lattices in the sense of Kneser. $\rightarrow$ Kneser's neighbor method gives elliptic fibrations