Gluing compact matrix quantum groups

Daniel Gromada, Saarland University

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TR 195 SYMBOLIC TOOLS

Outline

1. Gluing for matrix groups
2. Gluing for finitely generated groups
3. Quantum groups
4. Main theorem

Matrix groups and gluing

Example

- Consider S_n represented by permutation matrices, e.g.,

$$S_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ldots \right\}$$

- How to represent the direct product $S_n \times \mathbb{Z}_2$?

Example

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- How to represent the direct product $S_n \times \mathbb{Z}_2$?
- The canonical way: direct sum

$$\begin{aligned} S_3 \times \mathbb{Z}_2 &= \{ A \oplus r \mid A \in S_3,\ r = \pm 1 \} \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \dots \right\} \end{aligned}$$

Example

- Consider S_n represented by permutation matrices, e.g.,

$$S_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \dots \right\}$$

- How to represent the direct product $S_n \times \mathbb{Z}_2$?
- Alternative way: product

$$\begin{aligned} S_3 \times \mathbb{Z}_2 &= \{rA \,|\, A \in S_3,\ r = \pm 1\} \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \dots \right\} \end{aligned}$$

Glued direct product

- Let $G \subseteq \mathrm{GL}_n$, $H \subseteq \mathrm{GL}_m$ be matrix groups
- Their **direct product** $G \times H$ can be identified with

$$G \times H = \{A \oplus B \mid A \in G,\ B \in H\}$$

- We define their **glued direct product** as

$$G \tilde{\times} H = \{A \otimes B \mid A \in G,\ B \in H\}$$

- In particular, for the case $m = 1$:

$$G \tilde{\times} \mathbb{Z}_k = \{e^{2\pi i j/k} A \mid A \in G,\ j = 1, \dots, k\}, \qquad G \tilde{\times} \mathbb{T} = \{zA \mid A \in G,\ z \in \mathbb{T}\}$$

- Glued direct product is a quotient of the direct product (not always isomorphic!)
- $S_n \tilde{\times} \mathbb{Z}_k \simeq S_n \times \mathbb{Z}_k$
- $O_n \tilde{\times} \mathbb{Z}_2 = O_n$
- $O_n \tilde{\times} \mathbb{Z}_k \dots$ not isomorphic to a direct product in general

Gluing procedure

- Consider G represented by block-diagonal matrices, that is,

$$G = \left\{ \begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix} \right\}_{g \in G}$$

- We define the **glued version** of G to be

$$\tilde{G} := \{A_g \otimes B_g \mid g \in G\}.$$

- Example: glued product $G \mathbin{\tilde{\times}} H = \widetilde{G \times H}$

Gluing procedure

- Suppose now that the block B is one-dimensional, so

$$G = \left\{ \begin{pmatrix} A_g & 0 \\ 0 & z_g \end{pmatrix} \right\}_{g \in G}$$

- The **glued version** is then given by

$$\tilde{G} := \{ z_g A_g \mid g \in G \}.$$

- Example: glued product $G \tilde{\times} \mathbb{Z}_k = \widetilde{G \times \mathbb{Z}_k}$

Dual perspective:
Finitely generated groups

Direct product

- Consider two finitely-generated groups

$$\mathrm{B} = \langle \beta_1, \ldots, \beta_m \rangle, \qquad \Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$$

- We define the **direct product**

$$\mathrm{B} \times \Gamma = \{(\beta, \gamma) \mid \beta \in \mathrm{B},\ \gamma \in \Gamma\}$$

- We can identify $\beta = (\beta, e)$, $\gamma = (e, \gamma)$, hence $\beta\gamma = (\beta, \gamma) = \gamma\beta$
- In particular $\mathrm{B} \times \Gamma = \langle \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n \rangle$ – it is finitely generated
- We can associate matrices to the generating sets

$$b = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_m \end{pmatrix}, \qquad c = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix}$$

- The generating set of $\mathrm{B} \times \Gamma$ then corresponds to $b \oplus c = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$

Glued direct product

- Consider two finitely-generated groups

$$\mathrm{B} = \langle \beta_1, \ldots, \beta_m \rangle, \qquad \Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$$

- We define the **glued direct product**

$$\mathrm{B} \underset{\sim}{\times} \Gamma := \langle \beta_i \gamma_j \rangle_{i,j} \subseteq \mathrm{B} \times \Gamma$$

- Consider again

$$b = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_m \end{pmatrix}, \qquad c = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix}$$

- The generators of $\mathrm{B} \underset{\sim}{\times} \Gamma$ are the entries of $b \otimes c$

Free product

- Consider two finitely-generated groups

$$\mathrm{B} = \langle \beta_1, \ldots, \beta_m \rangle, \qquad \Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$$

- We define the **free product**

$$\mathrm{B} * \Gamma := \{\text{reduced words in generators } \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n\}$$

- In particular

$$\mathrm{B} * \Gamma = \langle \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_n \rangle \qquad \ldots b \oplus c$$

- We define the **glued free product**

$$\mathrm{B} \underset{\approx}{*} \Gamma := \langle \beta_i \gamma_j \rangle_{i,j} \subseteq \mathrm{B} * \Gamma \qquad \ldots b \otimes c$$

Free product

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- In particular

$$B * \Gamma = \langle \beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n \rangle \qquad \dots b \oplus c$$

- We define the **glued free product**

$$B \underset{\approx}{*} \Gamma := \langle \beta_i \gamma_j \rangle_{i,j} \subseteq B * \Gamma \qquad \dots b \otimes c$$

Gluing in general

- Consider $\Gamma = \langle \gamma_1, \dots, \gamma_n, \beta_1, \dots, \beta_m \rangle$
- We define its **glued version**

$$\tilde{\Gamma} := \langle \gamma_i \beta_j \rangle_{i,j} \subseteq \Gamma$$

Quantum groups

Quantum groups

Compact matrix groups

- Represented by some matrices

$$\begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$$

- Matrix entries commute
- Matrices may not commute

Fin. generated discrete groups

- Put generators in a matrix

$$\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix}$$

- Matrix entries may not commute
- Matrices commute

Compact matrix quantum groups = Fin. gen. discrete quan. groups

- Neither matrix entries nor matrices commute

Compact matrix quantum groups

- **Unitary CMQG** is a pair $G = (A, u)$, where

a) $A =: O(G)$ is a $*$-algebra

b) $u = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$

c) $u_{ij} \in A$

1. u_{ij} generate A
2. $uu^* = u^*u = \bar{u}u^t = u^t\bar{u} = 1_n$
3. $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ extends to a $*$-homomorphism $\Delta \colon A \to A \otimes A$

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- EXAMPLE: Compact matrix group G
  - u is the fundamental representation of G, i.e.
  - $u_{ij}\colon G \to \mathbb{C}$ mapping $g \mapsto g_{ij}$
  - $O(G)$:=polynomials in u_{ij}

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- EXAMPLE: **Free unitary quantum group U_n^+**

$$O(U_n^+) = *\text{-alg}(u_{ij},\ i,j = 1,\dots,n \mid uu^* = u^*u = \bar{u}u^t = u^t\bar{u} = 1_n)$$

- EXAMPLE: Unitary group U_n

$$O(U_n) = *\text{-alg}(u_{ij},\ i,j = 1,\dots,n \mid uu^* = u^*u = 1_n,\ u_{ij}u_{kl} = u_{kl}u_{ij})$$

Compact matrix quantum groups

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- EXAMPLE: **Free orthogonal quantum group O_n^+**

$$O(O_n^+) = *\text{-alg}(u_{ij},\ i,j = 1,\dots,n \mid uu^t = u^t u = 1_n,\ u_{ij} = u_{ij}^*)$$

- EXAMPLE: Orthogonal group O_n

$$O(O_n) = *\text{-alg}(u_{ij},\ i,j = 1,\dots,n \mid uu^t = u^t u = 1_n,\ u_{ij} = u_{ij}^*,\ u_{ij}u_{kl} = u_{kl}u_{ij})$$

Finitely generated discrete quantum groups

- **FGDQG** is a pair $\Gamma = (A, u)$, where

a) $A =: \mathbb{C}\Gamma$ is a $*$-algebra

b) $u = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$

c) $u_{ij} \in A$

1. u_{ij} generate A
2. $uu^* = u^*u = \bar{u}u^t = u^t\bar{u} = 1_n$
3. $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ extends to a $*$-homomorphism $\Delta \colon A \to A \otimes A$

- Example: Finitely generated group Γ
  - $\mathbb{C}\Gamma$ is the group algebra of Γ
  - $u = \mathrm{diag}(g_1, \ldots, g_n)$, where $g_1, \ldots, g_n$ are the generators of Γ

Products of quantum groups

- Consider CMQG=FGDQG (A, u), (B, v), we define the following:
- **Tensor product** $(A \otimes B, u \oplus v)$
- **Glued tensor product** $(C, u \otimes v)$, $C \subseteq A \otimes B$ generated by $u_{ij} v_{kl}$
- **Free product** $(A * B, u \oplus v)$
- **Glued free product** $(C, u \otimes v)$, $C \subseteq A * B$ generated by $u_{ij} v_{kl}$

Products of quantum groups

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- **Free product** $(A * B, u \oplus v)$
- **Glued free product** $(C, u \otimes v)$, $C \subseteq A * B$ generated by $u_{ij} v_{kl}$

Gluing quantum groups

- Consider $G = (A, u)$ with $u = v_1 \oplus v_2$
- We define its **glued version** $\tilde{G} = (\tilde{A}, \tilde{u})$, where
  - $\tilde{u} = v_1 \otimes v_2$
  - $\tilde{A} \subseteq A$ generated by matrix entrie of $\tilde{u}$

Main theorem

- THEOREM: Gluing defines a one-to-one correspondence between

$G \subseteq O_n^+ \hat{*} \hat{\mathbb{Z}}_2$ with degree of reflection two	$\longleftrightarrow$	$\tilde{G} \subseteq U_n^+$ invariant w.r.t. colour inversion

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- **degree of reflection two** = all relations in $O(G)$ are $\mathbb{Z}_2$-homogeneous
- **invariant w.r.t. color inversion** = there is an isomorphosm $u_{ij} \leftrightarrow u_{ij}^*$

Main theorem – discrete group version

- THEOREM: Gluing defines a one-to-one correspondence between

Γ quotient of $\mathbb{Z}_2^{*n} * \mathbb{Z}_2$, i.e.
$\Gamma = \langle a_1, \ldots, a_n, r \rangle$, $a_i^2 = e$, $r^2 = e$,
that is $\mathbb{Z}_2$-graded
(generators of degree one)

$\longleftrightarrow$

$\tilde{\Gamma}$ quotient of $\mathbb{Z}^{*n}$, i.e.
$\tilde{\Gamma} = \langle \tilde{a}_1, \ldots, \tilde{a}_n \rangle$, such that
$\tilde{a}_i \mapsto \tilde{a}_i^{-1}$ is a group
isomorphism

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Γ quotient of $\mathbb{Z}_2^{*n} * \mathbb{Z}_2$, i.e. $\Gamma = \langle a_1, \ldots, a_n, r \rangle$, $a_i^2 = e$, $r^2 = e$, that is $\mathbb{Z}_2$-graded (generators of degree one)	$\longleftrightarrow$	$\tilde{\Gamma}$ quotient of $\mathbb{Z}^{*n}$, i.e. $\tilde{\Gamma} = \langle \tilde{a}_1, \ldots, \tilde{a}_n \rangle$, such that $\tilde{a}_i \mapsto \tilde{a}_i^{-1}$ is a group isomorphism

- IDEA/PROOF: Direction $\rightarrow$ **gluing:** $\tilde{\Gamma}$ is a subgroup of Γ generated by

$$\tilde{a}_i := a_i r$$

We need to prove that $\tilde{\Gamma}$ has isomorphism $\tilde{a}_i \mapsto \tilde{a}_i^{-1}$.

This is indeed true: Conjugating by r defines an isomorphism on Γ

$$x \mapsto rxr$$

This isomorphism restricts to $\tilde{\Gamma}$ mapping

$$\tilde{a}_i \mapsto r\tilde{a}_i r = r a_i r^2 = r a_i = (a_i r)^{-1} = \tilde{a}_i^{-1}$$

Main theorem

- THEOREM: Gluing defines a one-to-one correspondence between

$G \subseteq O_n^+ \hat{*} \hat{\mathbb{Z}}_2$ with degree of reflection two	$\longleftrightarrow$	$\tilde{G} \subseteq U_n^+$ invariant w.r.t. colour inversion

- IDEA/PROOF: Direction $\to$ **gluing:** $O(\tilde{G})$ is a subalgebra of $O(G)$ generated by

$$\tilde{u}_{ij} := u_{ij} r$$

We need to prove that $O(\tilde{G})$ has isomorphism $\tilde{u}_{ij} \mapsto \tilde{u}_{ij}^*$.

This is indeed true: Conjugating by r defines an isomorphism on $O(G)$

$$x \mapsto rxr$$

This isomorphism restricts to $O(\tilde{G})$ mapping

$$\tilde{u}_{ij} \mapsto r\tilde{u}_{ij}r = ru_{ij}r^2 = ru_{ij} = (u_{ij}r)^* = \tilde{u}_{ij}^*$$

Main theorem – discrete group version

- THEOREM: Gluing defines a one-to-one correspondence between

Γ quotient of $\mathbb{Z}_2^{*n} * \mathbb{Z}_2$, i.e.
$\Gamma = \langle a_1, \ldots, a_n, r \rangle$, $a_i^2 = e$, $r^2 = e$
which is $\mathbb{Z}_2$-graded
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$\longleftrightarrow$

$\tilde{\Gamma}$ quotient of $\mathbb{Z}^{*n}$, i.e.
$\tilde{\Gamma} = \langle \tilde{a}_1, \ldots, \tilde{a}_n \rangle$, such that
$\tilde{a}_i \mapsto \tilde{a}_i^{-1}$ is a group
isomorphism

- IDEA/PROOF: Direction $\leftarrow$ **ungluing:**

  Consider some $\tilde{\Gamma}$

  Define $\tilde{N} \trianglelefteq \mathbb{Z}^{*n}$ such that $\tilde{\Gamma} = \mathbb{Z}^{*n}/\tilde{N}$

  Denote $\iota \colon \mathbb{Z}^{*n} \to \mathbb{Z}_2^{*n} * \mathbb{Z}_2$ the inclusion $\tilde{a}_i \mapsto a_i r$ (prove that it is indeed an injective homomorphism...)

  Define $N := \iota(\tilde{N})$ and $\Gamma := (\mathbb{Z}_2^{*n} * \mathbb{Z}_2)/N$

Main theorem

- THEOREM: Gluing defines a one-to-one correspondence between

$G \subseteq O_n^+ \hat{*} \hat{\mathbb{Z}}_2$ with degree of reflection two $\longleftrightarrow$ $\tilde{G} \subseteq U_n^+$ invariant w.r.t. colour inversion

- IDEA/PROOF: Direction $\leftarrow$ **ungluing:**

  Consider some $\tilde{G}$

  Define an ideal $\tilde{I} \subseteq \mathbb{C}\langle \tilde{x}_{ij}, \tilde{x}_{ij}^* \rangle$ such that $O(\tilde{G}) = \mathbb{C}\langle \tilde{x}_{ij}, \tilde{x}_{ij}^* \rangle / \tilde{I}$

  Denote $\iota \colon \mathbb{C}\langle \tilde{x}_{ij}, \tilde{x}_{ij}^* \rangle \to \mathbb{C}\langle x_{ij}, x_{ij}^* \rangle * \mathbb{C}\mathbb{Z}_2$ the inclusion $\tilde{x}_{ij} \mapsto x_{ij} r$ (prove that it is indeed an injective homomorphism...)

  Define $I := \iota(\tilde{I})$ and $O(G) := (\mathbb{C}\langle x_{ij}, x_{ij}^* \rangle * \mathbb{C}\mathbb{Z}_2)/I$

Main theorem – matrix group version

- Does not exist
- More precisely
  - gluing – works fine (matrix group $\to$ matrix group)
  - ungluing – does not work (matrix group may be unglued to a quantum group)
- EXAMPLE: The ungluing of U_n is not a group

  The commutativity in $O(U_n)$

$$\tilde{u}_{ij}\tilde{u}_{kl} = \tilde{u}_{kl}\tilde{u}_{ij}$$

  is unglued (recall $\tilde{u}_{ij} \mapsto u_{ij}r$) into

$$u_{ij}ru_{kl} = u_{kl}ru_{ij}$$

  Hence, the ungluing of U_n is a quantum subgroup of $O_n \hat{*} \hat{\mathbb{Z}}_2$ w.r.t. the weakenned commutativity above.

  If we require full commutativity, we obtain $O_n \times \hat{\mathbb{Z}}_2$, whose glued version is simply O_n.

Application: New $\mathbb{Z}_2$-extensions

- Take any $H \subseteq O_n^+$
- Construct $\tilde{G} := H \tilde{\times} \hat{\mathbb{Z}}_k \subseteq U_n^+$
- Compute the $\mathbb{Z}_2$-ungluing $G =: H \times_k \hat{\mathbb{Z}}_2$

H	v	
$H \times \hat{\mathbb{Z}}$	$v \oplus z$	$v_{ij} z = z v_{ij}$
$\tilde{G} := H \tilde{\times} \hat{\mathbb{Z}}$	$\tilde{u} := vz$	$\tilde{u}_{ij} \tilde{u}_{kl}^* = \tilde{u}_{ij}^* \tilde{u}_{kl}$
$G =: G \times_0 \hat{\mathbb{Z}}_2$	$u \oplus r$	$u_{ij} u_{kl} = r u_{ij} u_{kl} r$

- We can generalize this construction to define new products

$$G * H \begin{matrix} \supseteq \\ \supseteq \end{matrix} \begin{matrix} G \circledast H \\ G \circledast H \end{matrix} \begin{matrix} \supseteq \\ \supseteq \end{matrix} G \times_0 H \supseteq G \times_{2k} H \supseteq G \times_{2l} H \supseteq G \times_2 H = G \times H,$$