TALK: COUNTING RATIONAL POINTS ON TORIC VARIETIES VIA COX RINGS.

MARTA PIEROPAN

Let k be a number field, $\mathcal{O}_k$ its ring of integers and Cl_k its ideal class group.

Let X be a smooth, complete, split toric variety over k,
equivariant compactification of a torus U
defined by a fan Σ with rays $\rho_1, \dots, \rho_N$.

For all $i = 1, \dots, N$, let D_{ρ_i} be the torus invariant divisor associated to ρ_i.
Then $-K_X := \sum_{i=1}^N D_{\rho_i}$ is an anticanonical divisor of X.

Assume that $\mathcal{O}_X(-K_X)$ is generated by global sections,
let $\varphi : X \to \mathbb{P}_k^m$ be the induced morphism.

Let

$$H_{\mathbb{P}_k^m} : \mathbb{P}_k^m(k) \to \mathbb{R}_{\geq 0}, \qquad (y_0 : \cdots : y_m) \mapsto \prod_{\nu \in \Omega_k} \sup_{i=0,\dots,m} |y_i|_\nu$$

where Ω_k is the set of places of k and $|.|_\nu$ the norm associated to the place ν.
If $k = \mathbb{Q}$, then $H_{\mathbb{P}_k^m}(y_0 : \cdots : y_m) = \max_{i=0,\dots,m} |y_i|$ whenever $y_0, \dots, y_m \in \mathbb{Z}$ and $\gcd(y_0, \dots, y_m) = 1$. We note that we can always find such a representative for a $\mathbb{Q}$-rational point of $\mathbb{P}_k^m$. In this case the norm $|.|$ involved is the absolute value on $\mathbb{R}$.
We define $H : X(k) \to \mathbb{R}_{\geq 0}$ by $H = H_{\mathbb{P}_k^m} \circ \varphi$.

Manin's conjecture predicts that

$$N(B) := \#\{x \in U(k) : H(x) \leq B\} \sim C_{k,X} B (\log B)^{rk\,\mathrm{Pic}(X) - 1}, \quad B \to \infty.$$

How to approach the proof.

$\mathrm{Cox}(X) \cong k[x_1, \dots, x_N]$, graded by $\mathrm{Pic}(X)$: $\deg(x_i) = [D_{\rho_i}]$ in $\mathrm{Pic}(X)$.

Let Σ_{max} be the set of maximal cones of Σ.
For all $\sigma \in \Sigma_{max}$ we set $\underline{x}^\sigma = \prod_{1 \leq i \leq N, \rho_i \notin \sigma} x_i$.
Then

$$\mathcal{T} := \mathrm{Spec}(\mathrm{Cox}(X)) \setminus V(\{x^\sigma : \sigma \in \Sigma_{max}\})$$

is a universal torsor of X. In particular it is a torsor of X under $T_{NS} = \mathrm{Pic}(X)^* \cong \mathbb{G}_m^r$, where $r = rk\,\mathrm{Pic}(X)$.

Date: September 5, 2013.

$$\begin{array}{lll} T_{NS} \hookrightarrow \mathcal{T} \xrightarrow{\pi} X & & \text{over } k \text{ there is only one universal torsor} \\ \quad\quad\;\; \downarrow \quad\;\; \downarrow & & \\ \widetilde{T_{NS}} \hookrightarrow \widetilde{\mathcal{T}} \xrightarrow{\pi} \widetilde{X} & & \text{over } \mathcal{O}_k \text{ there are } H^1(\mathrm{Spec}(\mathcal{O}_k), \mathbb{G}_m^r) \cong Cl_k^r \text{ universal torsors.} \end{array}$$

We have that

$$X(k) = X(\mathcal{O}_k) = \bigsqcup_{[a]\in Cl_k^r} \pi({}_{\underline{\mathfrak{a}}}\widetilde{\mathcal{T}}(\mathcal{O}_k))$$

where the first equality holds because X is proper and ${}_{\underline{\mathfrak{a}}}\widetilde{\mathcal{T}}$ is a twist of $\widetilde{\mathcal{T}}$ with class $[a]$ in Cl_k^r.

For all $x \in {}_{\underline{\mathfrak{a}}}\widetilde{\mathcal{T}}(\mathcal{O}_k)$ we have $H(\pi(x)) = \frac{1}{N(a^{-K_X})} \prod_{\nu\in\Omega_\infty} \sup_{\sigma\in\Sigma_{max}} |\underline{x}^{-K_X(\sigma)}|_\nu$, where Ω_∞ is the set of the infinite places of k and $\underline{x}^{-K_X(\sigma)}$ are suitable monomials of degree $[-K_X]$ in $\mathrm{Cox}(X)$.

Let $V = \pi^{-1}(U)$, then

$$N(B) = \sum_{[a]\in Cl_k^r} \# \left(\{x \in V(k) \cap {}_{\underline{\mathfrak{a}}}\widetilde{\mathcal{T}}(\mathcal{O}_k) : H(\pi(x)) \le B\} / \widetilde{T_{NS}}(\mathcal{O}_k) \right).$$

$\widetilde{T_{NS}}(\mathcal{O}_k) \cong (\mathcal{O}_k^\times)^r \cong \mu_k^r \times U_k^r$, where μ_k is a finite group and U_k is torsion free.

Let Δ be a fundamental domain for the action of U_k^r on $V(k)$, then

$$N(B) = \frac{1}{(\#\mu_k)^r} \sum_{[a]\in Cl_k^r} \#\{x \in \Delta(k) \cap {}_{\underline{\mathfrak{a}}}\widetilde{\mathcal{T}}(\mathcal{O}_k) : H(\pi(x)) \le B\}.$$

Now we can proceed with a Möbius inversion and lattice point counting if the set $\{x \in \Delta(k) : H(x) \le B\}$ is bounded after embedding k into the product of its completions at the infinite places.

References

[1] Batyrev, V.V.; Tschinkel, Y. *Manin's conjecture for toric varieties.* J. Algebraic Geom. 7 (1998), no. 1, 15-53.

[2] Salberger, P. *Tamagawa measures on universal torsors and points of bounded height on Fano varieties.* Nombre et repartition de points de hauteur bornée (Paris,1996). Astérisque No. 251 (1998), 91-258.

[3] Schanuel, S.H. *Heights in number fields.* Bull. Soc. Math. France 107 (1979), no. 4, 433-449.

Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany

E-mail address: marta.pieropan@mathematik.uni-muenchen.de