Quasiprojectivity of toric varieties via convex continuous piecewiese linear functions

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Conjecture (Chevalley, 1957)

If X is a complete normal algebraic variety such that each finite subset of X is contained in some open affine subset of X then X is projective.

$$a(X) := \sup\{n \mid \text{each set of } n \text{ points on } X \text{ is contained in some open affine subset of } X\}.$$

$$\rho(X) := \dim\left(\left(\mathrm{CaDiv}(X)/\,\mathrm{CaDiv}^{\tau}(X)\right) \otimes_{\mathbb{Z}} \mathbb{R}\right)$$

Theorem (Kleiman, 1966)

If X is a complete $\mathbb{Q}$-factorial algebraic variety and $a(X) \geqslant 2\rho(X)$ then X is projective.

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If X is a complete $\mathbb{Q}$-factorial algebraic variety and $a(X) \geqslant 2\rho(X)$ then X is projective.

Problem

For every positive integer k *construct a complete normal algebraic variety with* $\rho(X) = 0$ *and* $a(X) = k$.

A cone is the subset $\mathbb{R}_{\geqslant 0}x_1 + \ldots + \mathbb{R}_{\geqslant 0}x_k$ of $\mathbb{R}^n$, where $x_1, \ldots, x_n \in \mathbb{Z}^n$.

The dual cone of a cone σ is the set $\sigma^\vee = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geqslant 0 \text{ for every } x \in \sigma\}$.

We use the semigroup $S_\sigma = \sigma^\vee \cap \mathbb{Z}^n$ to obtain an algebra $\mathbb{C}[S_\sigma]$ and a variety $X_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$.

A fan Δ in $\mathbb{R}^n$ is a finite set of cones in $\mathbb{R}^n$ such that:

1. each face of a cone from Δ is also in Δ,
2. the intersection of two cones from Δ is a face of each of them.

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If τ is a face of σ then $\sigma^\vee$ is a face of $\tau^\vee$, $\mathbb{C}[S_\sigma]$ is a subalgebra of $\mathbb{C}[S_\tau]$ and X_τ is an open subvariety of X_σ.

We construct the toric variety X_Δ by glueing together affine toric varieties X_σ for all $\sigma \in \Delta$.

$X_{\{0\}} = (\mathbb{C}^*)^n$ is an open subset of every X_σ.

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The map $y \mapsto \operatorname{div}(\chi^{-y})$ gives a surjection from $\mathbb{Z}^n$ onto $\operatorname{CaDiv}_T^0(X_\Delta)$ with kernel $\mathbb{Z}^n \cap |\Delta|^\perp$.

$D \in \operatorname{CaDiv}_T(X_\Delta)$ determines on each cone $\sigma \in \Delta$ an element $y_D(\sigma) \in \mathbb{Z}^n$ such that $D|_{X_\sigma} = \operatorname{div}\left(\chi^{-y_D(\sigma)}\right)|_{X_\sigma}$. It is determined uniquely up to addition of elements of $\sigma^\perp$.

Moreover if τ is a common face of $\sigma_1, \sigma_2 \in \Delta$ then $y_D(\sigma_1) - y_D(\sigma_2) \in \tau^\perp$.

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We associate with a divisor $D \in \mathrm{CaDiv}_T(X_\Delta)$ a piecewise linear function Ψ_D on the support $|\Delta|$ defined as $\Psi_D(x) = \langle x, y_D(\sigma)\rangle$ for $x \in \sigma$.

D is a principal divisor iff Ψ_D is a linear function.

$\Delta(1)$ – set of 1-dimensional cones in Δ

A divisor $D \in \mathrm{CaDiv}_T(X_\Delta)$ is uniquely determined by $y_D(\tau)$ for every $\tau \in \Delta(1)$.

A piecewise linear function Ψ on Δ is uniquely determined by the values $\Psi(x_\tau)$ for every $\tau \in \Delta(1)$ and any $x_\tau \in \tau \setminus \{0\}$.

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We call the function Ψ_D strictly convex if for every maximal cone $\sigma \in \Delta$ there is a linear function $L_D(\sigma)$ such that $L_D(\sigma)(x) = \Psi_D(x)$ for $x \in \sigma$ and $L_D(\sigma)(x) > \Psi_D(x)$ for $x \in |\Delta| \setminus \sigma$.

Theorem

$D \in \mathrm{CaDiv}_T(X_\Delta)$ is an ample divisor iff Ψ_D is strictly convex.

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Theorem

$D \in \mathrm{CaDiv}_T(X_\Delta)$ *is an ample divisor iff* Ψ_D *is strictly convex.*

Lemma

For a toric variety X_Δ we have: $a(X) \geqslant n$ iff for every $\sigma_1, \ldots, \sigma_n$ in Δ there exists a strictly convex piecewise linear function on $\bigcup_{i=1}^{n} \sigma_i$.

Theorem

For every $t \geqslant 5$ there is a complete normal toric variety X with $a(X) = t$ and $\rho(X) = 0$.

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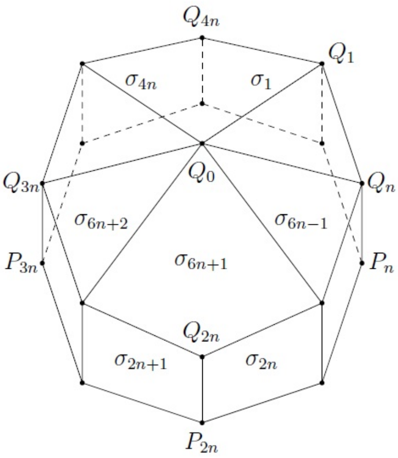
Theorem

For every $t \geqslant 5$ there is a complete normal toric variety X with $a(X) = t$ and $\rho(X) = 0$.

Let T be the convex polyhedron whose vertices are the points from the set $\mathcal{P} = \{P_1, \ldots, P_{4n}, Q_1, \ldots, Q_{4n}, Q_0\}$, where for $k = 1, \ldots, \frac{n}{2}$ we have:

$P_k = (nk + k^2, n^2 - k^2, -n^2)$,
$P_{\frac{n}{2}+k} = (n^2 - (\frac{n}{2} - k)^2, n(\frac{n}{2} - k) + (\frac{n}{2} - k)^2, -n^2)$,
$P_{n+k} = (n^2 - k^2, -nk - k^2, -n^2)$,
$P_{\frac{3n}{2}+k} = (n(\frac{n}{2} - k) + (\frac{n}{2} - k)^2, -n^2 + (\frac{n}{2} - k)^2, -n^2)$,
$P_{2n+k} = (-nk - k^2, -n^2 + k^2, -n^2)$,
$P_{\frac{5n}{2}+k} = (-n^2 + (\frac{n}{2} - k)^2, -n(\frac{n}{2} - k) - (\frac{n}{2} - k)^2, -n^2)$,
$P_{3n+k} = (-n^2 + k^2, nk + k^2, -n^2)$,
$P_{\frac{7n}{2}+k} = (-n(\frac{n}{2} - k) - (\frac{n}{2} - k)^2, n^2 - (\frac{n}{2} - k)^2, -n^2)$. Moreover:
$Q_k = P_k + (0, 0, 2n^2)$ dla $k = 1, \ldots, 2n - 1, 2n + 1, \ldots, 4n - 1$,
$Q_{2n} = (0, -n^2, n^2 - \frac{n^2}{n^2-1})$, $Q_{4n} = (0, n^2, n^2 - \frac{n^2}{n^2-1})$,
$Q_0 = (0, 0, 2n^2)$.



Polyhedron T for $n = 2$.

Let Δ' be the set of cones spanned over $\mathbb{R}_{\geqslant 0}$ by the faces, edges and vertices of T and $(0,0,0)$. We obtain a variety $X_{\Delta'}$.

We modify the fan Δ' by shifting the point P_{4n} to $(0, n^2, -n^2 - \varepsilon)$. We obtain the fan Δ. The variety X_Δ satisfies $a(X_\Delta) = 2n + 2$ and $\rho(X_\Delta) = 0$.

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Thank you for your attention.